MATH 548 Homework 3

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Section 5.7

Exercise 4

Using the binomial theorem $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, to expand $(x + y)^5$ to get $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$, and expand $(x + y)^6$ to get $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

Exercise 10

We will verbally prove this equality. Let the left hand side, $k\binom{n}{k}$, represent the number of ways you can choose k team members from n people where one team member is a captain. This is quite obviously $k\binom{n}{k}$ because you may pick k members in $\binom{n}{k}$ ways, and for each of those ways, you may choose any of the k members to be the captain, thus $k\binom{n}{k}$. Now let's consider the right hand side to realize that it represents the same situation: let's imagine that we are trying to, once again, construct teams of size k from n people, each of them having one captain. But this time, let's do the construction in a different way: choose one person out of all n people to be captain, and then construct all groups out of the remaining players and add that chosen captain to each. After the choice of that one person, it is obvious that we have only $n - 1$ people to choose from, and we must choose only $k - 1$ players for each group (as we are going to add the captain in later), so we have $\binom{n-1}{k-1}$ choices. But we must repeat this process for each of the n possible captains, so we result with $n\binom{n-1}{k-1}$ choices. Because the set we were constructing was the exact same in both cases, we claim we have proved the equality.

Exercise 15

Consider Theorem 5.2.2, which states $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all x. And let's take the derivative of both sides of the equation, which results in $n(1 + x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^k$. Note that the index has changed on the summation on the right hand side because the case where $k = 0$ causes the first term to be a constant, which has derivative 0. Now remember that this statement holds for all x, and so it should hold for the case where $x = -1$, which is $0 = \sum_{k=1}^{n} k\binom{n}{k}(-1)^k$. Notice now that the right hand side of this equation is equivalent to our stated problem, and so we have shown that it equals zero.

Exercise 23b

First, let us tackle the number of possible walks between her home and her best friend's home. We know that she must walk exactly 9 blocks to reach her friend's house (because she always takes a shortest route), and so at each junction, she could go either east or north, but she must go north a maximum of 5 blocks and east a maximum of 4 blocks, so we can imagine this problem as all the different arrangements of letters in a 9 letter word containing 5 'N's and 4 'E's, (for example "NNNENENEE"). But this is equivalent to all the possible ways you can place 5 'N's in 9 spots or 4 'E's in 9 spots, which is simply $\binom{9}{4}$. We apply the same reasoning to the remaining trip from her friend's house to school, which is 6 blocks east and 9 blocks north, which is $\binom{15}{6}$ different ways of walking. And because for each way she may walk to her friend's house, she may walk any way from her friend's house to school, we must multiply the two to get all the possible paths, so

$$\binom{15}{6}\binom{9}{4} = 630630$$

Section 6.7

Exercise 1

We may use the inclusion-exclusion principle to solve this problem, which is applied by first starting with all 10,000 integers, and then subtracting all of the integers that are divisible by 4,5, or 6. That is, $10,000 - \frac{10,000}{4} - \frac{10,000}{5} - \frac{10,000}{6}$. But then we realize that we overcompensated in our subtraction (that is, we double counted some integers), such as those divisible by both 5 and 6. So in order to account for those, we add back in all those values that we double counted in the subtraction, which should be all of those integers divisible by 30, 20, and 12 (12 representing all of those values divisible by both 4 and 6, as it is the least common multiple of the two). Then, finally, we realize that we have added twice all of those values that are divisible by 4,5, and 6, so we subtract all of those. Resulting in

$$10,000 - \frac{10,000}{4} - \frac{10,000}{5} - \frac{10,000}{6} + \frac{10,000}{30} + \frac{10,000}{20} + \frac{10,000}{12} - \frac{10,000}{60} = 5334$$

integers out of the first 10,000 are not divisible by 4,5, or 6.

Exercise 15

b) To find all of the combinations where at least one gentlemen receives his own hat, we may find all of the ways the hats may be distributed amongst the gentlemen, and then subtract all of those ways where no gentleman gentleman received his hat. That is, $n! - D_7$ where D_7 denotes the full derangement (because the number of gentlemen is 7). So, the result is

$$7! - 7!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}) = 5040 - 1854 = 3186$$

c) To find all the ways at least two gentlemen receive their own hat, we have to take all the ways the hats can be dealt, and then subtract all of the ways where no gentlemen receives his hat (determined in part b to be D_7) and also subtract all of the ways where one gentlemen receives his hat. We determine this to be $7D_6$, because if we consider the scenario where the first man receives his hat, there are D_6 possible ways for all of the other men to not receive their hat. But because we can choose any of the 7 men to have received their hat, we have $7D_6$. Thus, the result is

$$n! - D_7 - D_6 = 2921$$