Lecture 17:
Suffix Arrays

Not in Book

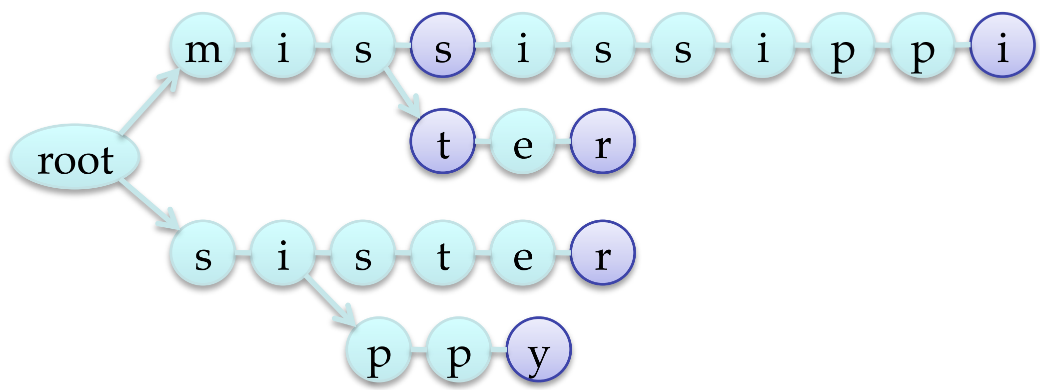
Homework #3
will be posted tonight

Searching Sequence Review

- Searching for a string of length *m* in a text of length n
- Indexing strings with trees
  - Keyword tree, O(m) search independent of number of keywords
  - Suffix tree O(n) construction, O(m) search
- Suffix Arrays: a practical alternative to Suffix tree search time: O(logn)
- Burrows-Wheeler transform, back to O(m)

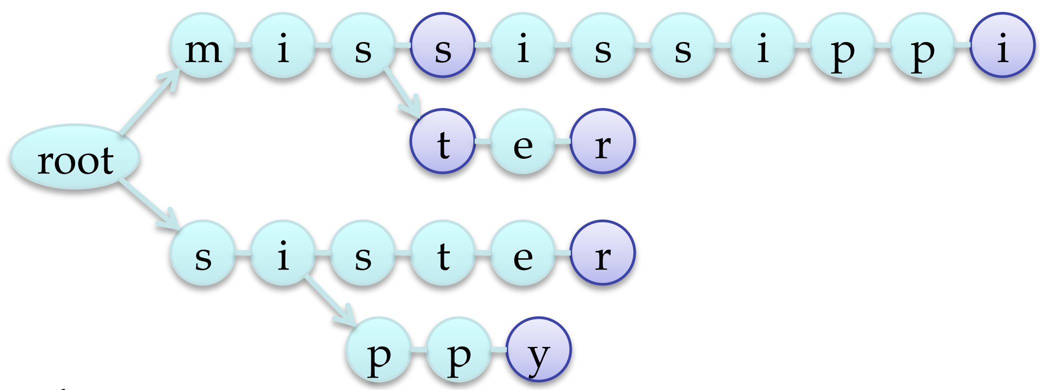
Keyword Tree

- A tree for representing a “dictionary” of terms
- Merges common prefixes into a single path
- Example:
  - miss
  - mississippi
  - mist
  - mister
  - sister
  - sippy



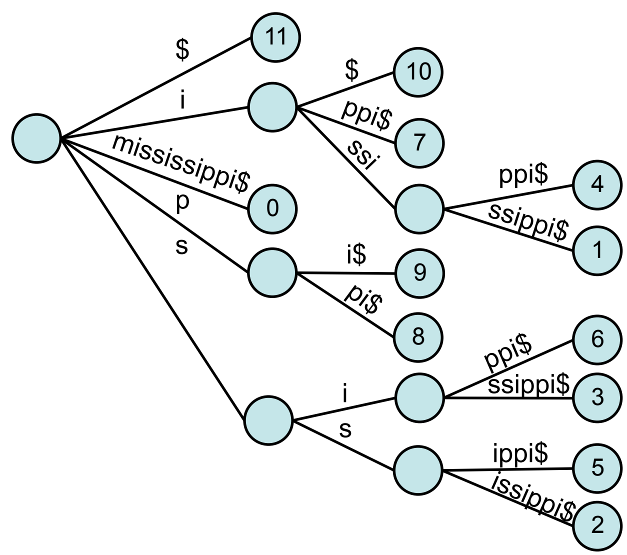
Keyword Tree

- Queries supported:
  Does keyword, k, appear in my text?
  - missstep
  - sip
- Searching via "Threading"
- Useful for spell checking, but hashing is preferred
- Not good for how many words contain "sis"



Recall Suffix Trees

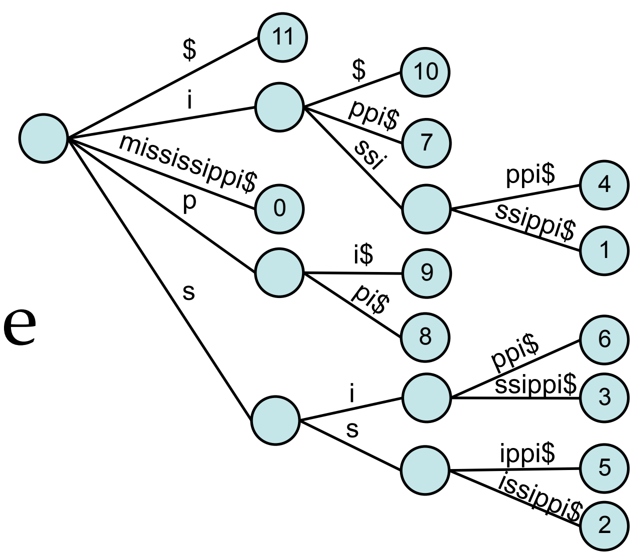
- A compressed keyword tree of suffixes from a given sequence
- Leaf nodes are labeled by the starting location of the suffix that terminates there
- Note that we now add an end-of-string character '$'



0. mississippi$
1. ississippi$
2. ssissippi$
3. sissippi$
4. issippi$
5. ssippi$
6. sippi$
7. ippi$
8. ppi$
9. pi$
10. i$
11. $

Suffix Tree Features

- How many leaves in a sequence of length m? O(m)
- How many nodes? (assume an alphabet of k characters) O(m)
- Given a suffix tree for a sequence. How long to determine if a pattern of length n occurs in the sequence? O(n)



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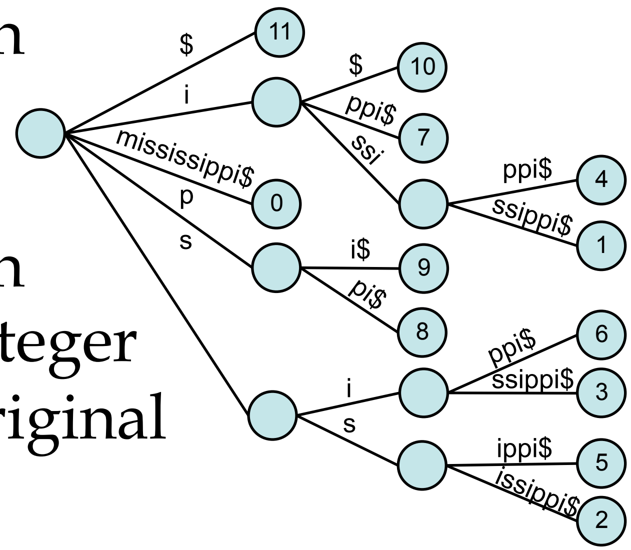
Suffix Tree Features

- How much storage?
  - Just for the edge strings $O(n^2)$
  - Trick: Rather than storing an actual string at each edge, we can instead store 2 integer offsets into the original text



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- In practice the storage overhead of Suffix Trees is too high, O(n) vertices with data and O(n) edges with associated data

Suffix Tree Properties

- There exists a depth-first traversal that corresponds to lexigraphical ordering (alphabetizing) all suffixes

11. $
10. i$
7. ippi$
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8. ppi$
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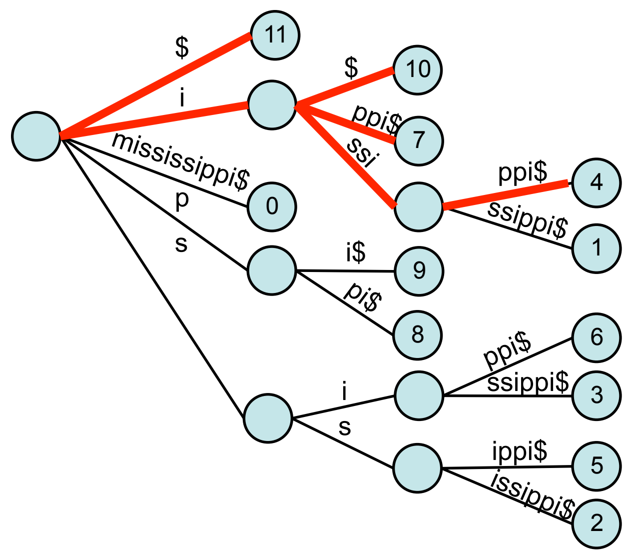
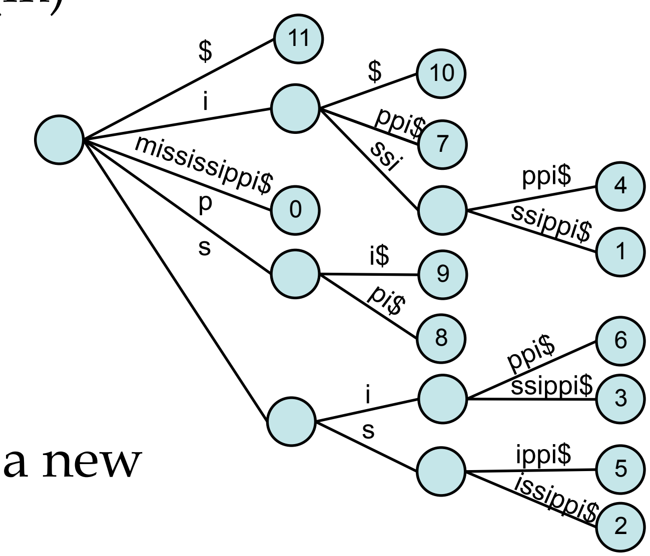


Suffix Tree Construction

- One could exploit this property to construct a Suffix Tree
  - Make a list of all suffixes: O(m)
  - Sort them: $O(m^2 \log m)$
  - Traverse the list from beginning to end while threading each suffix into the tree created so far, when the suffix deviates from a known path in the tree, add a new node with a path to a leaf.



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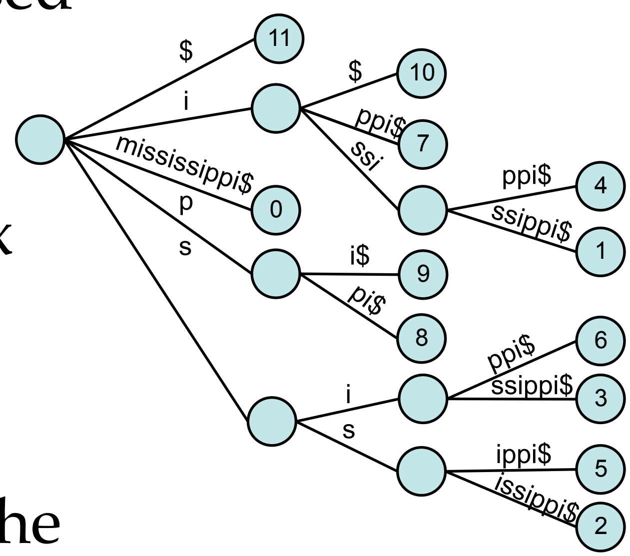
- ☹ Slower than the O(m) Ukkonen algorithm given last time

Saving space

- Sorting however did capture important aspects of the suffix trees structure
- A sorted list of tree-path traversals, our sorted list, can be considered a "compressed" version of a suffix tree.
- Save only the index to the beginning of each suffix

  11, 10, 7, 4, 1, 0, 9, 8, 6, 3, 5, 2



- Key: Argsort(text): returns the indices of the sorted elements of a text

Argsort

- One of the smallest Python functions yet:

```
def argsort(text):
    return sorted(range(len(text)), cmp=lambda i,j: -1 if text[i:] < text[j:] else 1)

print argsort("mississippi$")
```

```
$ python suffixarray.py
[11, 10, 7, 4, 1, 0, 9, 8, 6, 3, 5, 2]
```

- What types of queries can be made from this "compressed" form of a suffix tree
- We call this a "Suffix Array"

Suffix Array Queries

- Has similar capabilities to a Suffix Tree
- Does 'sip' occur in "mississippi"?
- How many times does 'is' occur?
- How many 'i''s?
- What is the longest repeated subsequence?
- Given a *suffix array* for a sequence. How long to determine if a pattern of length n occurs in the sequence? O(n log$m$)

11. $
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Searching Suffix Arrays

- Separate functions for finding the first and last occurrence of a pattern via binary search

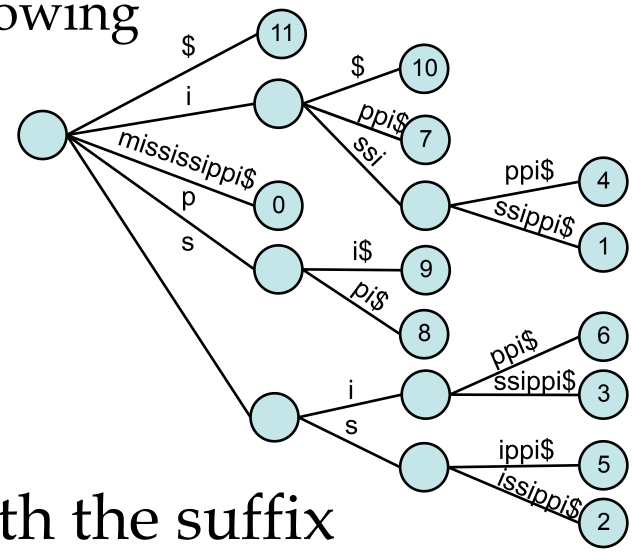
```
def findFirst(pattern, text, sfa):
    """ Finds the index of the first occurrence of pattern in the suffix array """
    hi = len(text)
    lo = 0
    while (lo < hi):
        mid = (lo+hi)//2
        if (pattern > text[sfa[mid]:]):
            lo = mid + 1
        else:
            hi = mid
    return lo

def findLast(pattern, text, sfa):
    """ Finds the index of the last occurrence of pattern in the suffix array """
    hi = len(text)
    lo = 0
    m = len(pattern)
    while (lo < hi):
        mid = (lo+hi)//2
        i = sfa[mid]
        if (pattern >= text[i:i+m]):
            lo = mid + 1
        else:
            hi = mid
    return lo-1
```

Augmenting Suffix Arrays

- It is possible to augment a suffix array to facilitate converting it into a suffix tree
- Longest Common Prefix, (lcp)
  - Note than branches, and, hence, interior nodes if needed are introduced immediately following a shared prefix of two adjacent suffix array entries

$	lcp = 0
i$	lcp = 1
ippi$	lcp = 1
issipi$	lcp = 4
ississippi$	lcp = 0
mississippi$	lcp = 0



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- If we store the lcp along with the suffix array it becomes a trivial matter to reconstruct and traverse the corresponding Suffix Array

Other Data Structures

- There is another trick for finding patterns in a text string, it comes from a rather odd remapping of the original text called a "Burrows-Wheeler Transform" or BWT.
- BWTs have a long history. They were invented back in the 1980s as a technique for improving lossless compression. BWTs have recently been rediscovered and used for DNA sequence alignments. Most notably by the Bowtie and BWA programs for sequence alignments.

String Rotation

- Before describing the BWT, we need to define the notion of Rotating a string. The idea is simple, a rotation of *i* moves the prefix_i, to the string's end making it a suffix.

Rotate("tarheel$", 3) → "heel$tar"

Rotate("tarheel$", 7) → "$tarheel"

Rotate("tarheel$", 1) → "arheel$t"

BWT Algorithm

BWT (string text)

table$_i$ = Rotate(text, i) for i = 0..len(text)-1

sort table alphabetically

return (last column of the table)

```
tarheel$          $tarheel
arheel$t          arheel$t
rheel$ta          eel$tarh
heel$tar          el$tarhe
eel$tarh          heel$tar
el$tarhe          l$tarhee
l$tarhee          rheel$ta
$tarheel          tarheel$
```

BTW(“tarheels$”) = “ltherea$”

BWT in Python

- Once again, this is one of the simpler algorithms that we've seen

```
def BWT(s):
    # create a table, with rows of all possible rotations of s
    rotation = [s[i:] + s[:i] for i in xrange(len(s))]
    # sort rows alphabetically
    rotation.sort()
    # return (last column of the table)
    return "".join([r[-1] for r in rotation])
```

- Input string of length *m*, output a messed up string of length *m*

Inverse of BWT

- A property of a transform is that there is no information loss and they are invertible.

```
inverseBWT(string s)
    add s as the first column of a table strings
    repeat length(s)-1 times:
        sort rows of the table alphabetically
        add s as the first column of the table
    return (row that ends with the 'EOF' character)
```

l	l$	l$t	l$ta	l$tar	l$tarh	l$tarhe	l$tarhee
t	ta	tar	tarh	tarhe	tarhee	tarheel	$\boxed{\texttt{tarheel\$}}$
h	he	hee	heel	heel$	heel$t	heel$ta	heel$tar
e	ee	eel	eel$	eel$t	eel$ta	eel$tar	eel$tarh
r	rh	rhe	rhee	rheel	rheel$	rheel$t	rheel$ta
e	el	el$	el$t	el$ta	el$tar	el$tarh	el$tarhe
a	ar	arh	arhe	arhee	arheel	arheel$	arheel$t
$	$t	$ta	$tar	$tarh	$tarhe	$tarhee	$tarheel

Inverse BTW in Python

- A slightly more complicated routine

```
def inverseBWT(s):
    # initialize table from s
    table = [c for c in s]
    # repeat length(s) - 1 times
    for j in xrange(len(s)-1):
        # sort rows of the table alphabetically
        table.sort()
        # insert s as the first column
        table = [s[i]+table[i] for i in xrange(len(s))]
    # return (row that ends with the 'EOS' character)
    return table[[r[-1] for r in table].index('$')]
```

BWT advantages

- A BWT is smaller than a suffix array
  - A BWT requires 2 m bits. As many bits as needed to represent a character in the alphabet, $log_2(4) = 2$, times the length of the string m. With compression you can do even better.
  - A suffix array requires $m \log_2(m)$ bits, as many bits as needed to represent an index into the string, $\log_2(m)$, times the number of suffixes. Does not compress well
- A BWT is faster than a suffix array
  - BWT O(n) search
  - Suffix array O(n log n)

Next Time

- The details of search using BWT's (get used to going backwards)
- FM indices
- Sampled FM indices